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ON THE UNCOUPLED SUPERPOSITION APPROXIMATION FOR COMBINED CONDUCTION–RADIATION THROUGH INFRARED RADIATING GASES

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NOMENCLATURE

D, band width parameter;
E_ν, Planck radiosity at wavenumber ν;
F, dimensionless energy flux, $F = q_t/D[E_{\nu_1} - E_{\nu_2}]$;
h, parallel plate spacing;
K_n, dimensionless transmission function for $n = 2$ and absorption function for $n = 3$;
k, thermal conductivity;
N, conduction–radiation parameter for gray radiation, $N = k\kappa/4\sigma\bar{T}^3$;
Q_g, dimensionless energy flux, equation (3);
Q_t, dimensionless energy flux, $Q_t = q_t h/k(T_1 - T_2)$;
q_t, total energy flux;
S, integrated absorption-band intensity;
T, temperature;
$\bar{T}$, average or reference temperature;
T_n, boundary temperature, $n = 1, 2$;
y, spatial coordinate.

Greek symbols

η, dimensionless coordinate, y/h;
θ, dimensionless temperature $(T - T_2)/(T_1 - T_2)$;
κ, Gray absorption coefficient;
ν, spectral radiation wavenumber;
ρ_a, absorbing species density;
σ, Stefan–Boltzmann constant;
τ_h, optical depth, $\rho_a \kappa h$ for a gray medium, $\rho_a Sh/D$ for band absorbing medium;
ϕ, dimensionless emissive power $(E_\nu - E_{\nu_2})/(E_{\nu_1} - E_{\nu_2})$;
ψ, conduction–radiation parameter for band absorption, $\psi = k(\rho_a S/D)/2(\partial E_\nu/\partial T)_{T=\bar{T}} D$.

INTRODUCTION

IT IS a well-known result that for a gray medium confined between infinite, parallel, plane walls at different temperatures, the total energy transfer by conduction and radiation in both the optically thin and optically thick limits, is simply the sum of the independent contributions of conduction alone and radiation alone [1]. No such general result is available for intermediate optical depths but it has been found in numerous comparisons [2–6] with results of fully coupled analyses that the maximum error suffered is less than 10 per cent for a gray medium and black boundaries. For sectionally gray media in which some region of the spectrum is totally transparent to thermal radiation, the error is generally less due to the fact that the total energy transfer consists of a sum of terms of which the dominant one may be evaluated exactly. The only terms being approximated are thermal conduction and total emission from the medium transmitted to the boundary. The remaining component of the energy flux consists of surface to surface transfer attenuated by absorption within the medium. This term may be evaluated without reference to the solution of the energy equation and thus reduces the resultant error in the total heat transfer. When there are transparent spectral regions this last term may become relatively large compared to the others reducing the error even further.

It thus seems to be well established that uncoupled superposition is a good approximation of exact results for gray or sectionally gray media bounded by black surfaces. Cess [1,4], however, points out the failure of the method for highly reflective surfaces.

Conceding that good results may be obtained for gray media it is natural to inquire whether or not the approximation might be useful for infrared energy transfer through absorbing–emitting molecular gases. The most salient feature of such gases is the pronounced spectral variation of the absorption coefficient [7,8]. Moreover, it has been shown that gray approximations are inadequate and lead to erroneous results when applied to these gases [9, 10] and as a result, one must be hesitant about transferring results of gray analyses to new situations.

Until recently, published results from analyses of combined conduction–radiation in infrared radiating gases have not been extensive enough for an evaluation of the superposition method but even if they had been the necessary radiative equilibrium solutions were not available at the time. Crosbie and Viskanta [11] have provided the latter for a number of absorption-band spectral profiles, one of which is representative of observed total absorption characteristics of most absorbing atmospheric and combustion gases. The model is the exponential band profile first proposed by Edwards and Menard [12]. This band profile has been employed by Nelson and Edwards [10] for extensive calculations of heat conduction coupled with radiative transfer employing a linearized Planck function. The results of this investigation and those of reference [11] may be used to evaluate the uncoupled superposition method applied to i.r. radiating gases.

ENERGY FLUX EQUATIONS

The total energy transfer rate per unit area through a planar, single band, absorbing–emitting medium whose black boundaries are maintained at different temperature T_1 and T_2 may be expressed in dimensionless form as

$$F = \frac{q_t}{D[E_{\nu_1} - E_{\nu_2}]} = -\frac{k}{h}\frac{[T_1 - T_2]}{D[E_{\nu_1} - E_{\nu_2}]}\frac{d\theta}{d\eta}\bigg|_{\eta=1} + \frac{\sigma[T_1^4 - T_2^4]}{D[E_{\nu_1} - E_{\nu_2}]} + 2K_3(\tau_h) + 2\tau_h\int_0^1 \phi(\eta') K_2[\tau_h(1-\eta')]\,d\eta' \quad (1)$$

where the narrow band approximation has been assumed, where properties have been taken constant and all symbols are as defined in the nomenclature. Alternative definitions of the K_n functions are available in [13–14]. Expressions for various band models may be found in [10, 14–16]. For

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linearized Planck radiosity the above is conveniently expressed somewhat differently as [10].

$$Q_t = \frac{q_t h}{k(T_1 - T_2)} = -\left.\frac{d\theta}{d\eta}\right|_{\eta=1} + \frac{4\sigma \bar{T}^3 h}{k} + \frac{\tau_h}{2\psi} \times \left\{2K_3(\tau_h) + 2\tau_h \int_0^1 \theta(\eta')K_2[\tau_h(1-\eta')]\,d\eta'\right\}. \quad (2)$$

Ordinarily the dimensionless temperature θ and Planck function ϕ must be obtained from the solution of an integro-differential, two-point, boundary value problem [5, 6, 10]. The uncoupled superposition approximation suggests that instead of solving the integro-differential equation the dimensionless energy fluxes F and Q may be approximated by putting $-d\theta/d\eta = 1$, $\phi = \theta = \phi^*$ where ϕ^* is the dimensionless emissive power for nonlinear radiation or the dimensionless temperature for linearized radiation obtained from the solution of the integral equation of radiative equilibrium [11].

Note that in equations (1) and (2) the second and third terms are independent of ϕ (or θ), and, therefore, can be quite easily obtained for any specific situation. Furthermore, the third term is always negative and when the second term is excluded results in the remaining terms passing through zero as a function of optical depth. Since the approximations generally do not possess the same zero this leads to the misleading result that at one point the percentage error in the approximation is infinite. As a result, the comparison of coupled and uncoupled solutions will be based upon the first and last terms of equation (2). This equation has been chosen since no results of nonlinear radiation appropriate to equation (1) exist, while [10] gives results for equation (2). The results of [11] are appropriate for ϕ^*. The comparison to follow is thus based upon

$$Q_g = -\left.\frac{d\theta}{d\eta}\right|_{\eta=1} + \frac{\tau_h^2}{\psi}\int_0^1 \theta(\eta')K_2[\tau_h(1-\eta')]\,d\eta' \quad (3)$$

and

$$Q_g^* = 1 + \frac{\tau_h}{2\psi} Q_e^* \quad (4)$$

where

$$Q_e^* = 2\tau_h \int_0^1 \phi^*(\eta')K_2[\tau_h(1-\eta')]\,d\eta'. \quad (5)$$

The value of Q_e^* was obtained from the results of [11]. Using their nomenclature, it is $Q_e^* = Q - 2K_3(\tau_h)$. Results for Q_g from [10] were obtained numerically by the Galerkin method to at least four significant figures.

RESULTS AND DISCUSSION

The comparison based upon equations (3) and (4) is more stringent than those usually given for gray or gray band media [5, 6] since all terms independent of the temperature distribution are excluded. As a result the error in equation (4) as compared with (3) will be larger, perhaps in some cases substantially so, than the error suffered in calculating the approximation of equation (2).

The comparisons which follow are based upon the percent error

$$\Sigma = 100(Q_g^* - Q_g)/Q_g. \quad (6)$$

Results for an exponential band are given in Table 1. Evidently the superposition method can be expected to give good results for total energy flux for i.r.-radiating, planar gases when linearization is appropriate and when only a single absorption–emission band is of importance. These two qualifications naturally lead to the question of how much the above conclusion might be altered by nonlinear radiative transfer and/or multiple nonoverlapping absorption–emission bands.

Table 1. Percent error in the superposition approximation for linearized radiation and an exponential band

τ_h	Conduction–radiation parameter, ψ 10^{-2}	10^{-1}	1	10	10^2
10^{-1}	−8·480	−1·301	−0·139	−0·014	−0·001
1	−8·236	−12·163	−4·773	−0·638	−0·066
10	0·477	−1·961	−6·374	−4·458	−0·758
10^2	4·363	4·051	1·574	−3·327	−3·158
10^3	4·317	4·313	4·032	1·718	−3·364

To somewhat gauge the nonlinear effect the gray band results of [10] may be compared to the nonlinear, gray medium results of [5]. Some simple manipulations of these later results were necessary to extract that part corresponding to the nonlinear equivalents of equations (3) and (4)*. Also, the dimensionless conduction–radiation parameter, N, defined for a gray medium in [5] is mathematically equivalent to 2ψ and when this substitution is made linearized gray medium results are mathematically equivalent to gray, narrow-band results from equations (3) and (4).

Table 2 compares the percent error for linearized and nonlinear radiative transfer for the range of parameters available from [5, 6, 10]. The nonlinear results are for a cold wall to hot wall temperature ratio of 0·5. The results of Cess [4] indicate the error is relatively insensitive to wall temperature ratios as small as 0·1. For the most part the nonlinear results have comparable errors to those for linearized radiation and the differences are not especially significant. Presuming nonlinearities have a similar effect on media with different absorption–emission characteristics, it appears that superposition errors should be of the same magnitude as those of Table 1 even when linearization is not permissible. It is clear on comparison of Tables 1 and 2 that gray-band errors are somewhat smaller than those for an exponential band.

Table 2. A comparison of percent error in the superposition approximation for linearized and nonlinear gray radiative transfer

τ_h	Linearized radiation, Conduction–radiation parameter, ψ 10^{-2}	10^{-1}	1	10
10^{-1}	−7·596	−1·136	−0·119	−0·012
1	−7·125	−8·782	−2·523	−0·302
10	−2·624	−2·934	−1·747	−0·299
	Nonlinear radiation $2{\cdot}5 \times 10^{-3}$	$2{\cdot}5 \times 10^{-2}$	$2{\cdot}5 \times 10^{-1}$	2·5
10^{-1}	—	−1·93	−0·212	−0·021
1	−6·016†	−9·037	−3·786	−0·515
10	—	−1·702	−1·965	−0·446

† From reference [6].

The matter of multiple absorption–emission bands must be considered unresolved. There have been no investigations to determine, for example, if the total radiative flux from the solution for a multi-band medium in radiative equilibrium could be approximated as the superposition of components resulting from the solution for the radiative equilibrium flux due to each band taken individually. This procedure would have to be used at present since only single band radiative equilibrium solutions are currently available.

Finally, it might be noted that an alternative approximation of the type presented in [5] does not appear to offer any special advantage in the present case, maximum errors being of the same magnitude.

* The nonlinear equivalent of equation (3) would have the function ϕ replace θ in the integral term.

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LETTERS TO THE EDITORS

COMMENTS ON THE PAPER "A THEORETICAL SOLUTION OF THE LOCKHART AND MARTINELLI FLOW MODEL FOR CALCULATING TWO-PHASE FLOW PRESSURE DROP AND HOLD-UP"

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IN THE paper [1] which appeared recently in this journal, there are a few errors.

Equation (15) of the peper giving the value of X as

$$X = \left[\left(\frac{\Delta p}{\Delta L}\right)_L^+ \Big/ \left(\frac{\Delta p}{\Delta L}\right)_G^+\right]^{0\cdot 5} = \left[\frac{M_L^{*(1-0\cdot 5n)}}{M_G^{*(1-0\cdot 5m)}}\right]\left[\frac{\eta_L^{0\cdot 5n}}{\eta_G^{0\cdot 5m}}\right]\left[\frac{\rho_L}{\rho_G}\right]^{-0\cdot 5}\left[\frac{C_L}{C_G}\right]^{0\cdot 5}$$

is not correct as is obvious because the r.h.s. can be dimensionless only for $n = m$. The correct expression for X is,

$$X = \left[\frac{M_L^{*(1-0\cdot 5n)}}{M_G^{*(1-0\cdot 5m)}}\right]\left[\frac{\eta_L^{0\cdot 5n}}{\eta_G^{0\cdot 5m}}\right]\left[\frac{\rho_L}{\rho_G}\right]^{-0\cdot 5}\left[\frac{C_L}{C_G}\right]^{0\cdot 5}\left[\frac{\pi}{4}D\right]^{0\cdot 5(n-m)}.$$

Also equation (21) giving the value of U_G is not correct. The equation should read as

$$U_G = 2R\left[\arccos\left(1-\frac{H}{R}\right)\right] = 2R\tilde{U}_G.$$

Because of this the curve for $\tilde{U}_G$ in Fig. 4 is displaced upwards. The two curves of $\tilde{U}_G$ and $\tilde{U}_L$ will intersect at

$$X = 1 \quad \text{for} \quad n = m$$

and

$$X = \pi^{0\cdot 5(n-m)} \quad \text{for} \quad n \neq m.$$

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